

# Minimal Valid Inequalities for Integer Constraints

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Dedicated to George Nemhauser for his 70th birthday

## Abstract

In this paper we consider a semi-infinite relaxation of mixed integer linear programs. We show that minimal valid inequalities for this relaxation correspond to maximal lattice-free convex sets, and that they arise from nonnegative, piecewise linear, positively homogeneous, convex functions.

## 1 Introduction

Consider a mixed integer linear program (IP)

$$\begin{aligned} \min \quad & cx \\ Ax \quad &= b \\ x_j \quad &\in \mathbb{Z} \quad \text{for } j = 1, \dots, p, \\ x_j \quad &\geq 0 \quad \text{for } j = 1, \dots, n, \end{aligned}$$

where  $p \leq n$ , the matrix  $A \in \mathbb{Q}^{m \times n}$ , the row vector  $c \in \mathbb{Q}^n$ , the column vector  $b \in \mathbb{Q}^m$  are data, and  $x \in \mathbb{R}^n$  is a column vector of variables. We assume that  $A$  has full row rank  $m$ .

A common approach to solve (IP) is to first solve the linear programming relaxation (LP) obtained by ignoring the integrality restrictions on  $x$ . Consider the corresponding optimal simplex tableau, where  $B$  and  $J$  denote the sets of basic and nonbasic variables respectively.

$$x_i = f_i + \sum_{j \in J} r^j x_j \quad \text{for } i \in B. \quad (1)$$

We have  $f \geq 0$ . If  $f_i \in \mathbb{Z}$  for all  $i \in B \cap \{1, \dots, p\}$ , then the basic solution  $x_i = f_i$  for all  $i \in B$  and  $x_i = 0$  otherwise, is an optimal solution of (IP). On the other hand, if  $f_i \notin \mathbb{Z}$  for

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some  $i \in B \cap \{1, \dots, p\}$ , the above basic solution is not feasible to (IP) and one may want to generate one or several cutting planes that cut it off while preserving all the feasible solutions of (IP). Different strategies have been proposed for generating cutting planes. For example, Balas [2] introduced *intersection cuts* obtained by intersecting the rays  $f + \alpha r^j$ ,  $\alpha \geq 0$ , with a convex set whose interior contains  $f$  but no integral point. Most general purpose cutting planes used in state-of-the-art solvers are obtained by generating a linear combination of the original constraints  $Ax = b$ , and by applying integrality arguments to the resulting equation (Gomory's Mixed Integer Cuts [12], MIR inequalities [17] and split cuts [6] are examples). Recently, there has been an interest in cutting planes that cannot be deduced from a single equation, but can be deduced by integrality arguments involving two equations (Dey and Richard [8], Andersen, Louveaux, Weismantel and Wolsey [1], Gomory [15]).

Gomory [13] introduced the *corner polyhedron*, obtained from the constraints of (IP) by dropping the nonnegativity restriction on all the basic variables  $x_i$ ,  $i \in B$ , in (1). Note that we can now drop the constraints  $x_i = f_i + \sum_{j \in J} r^j x_j$  for  $i \in B \cap \{p+1, \dots, n\}$  since these variables  $x_i$  only appear in one equation and no other constraint. Interestingly, the cutting planes mentioned above are valid not only for (IP) but also for the corner polyhedron. In this paper, we make one further relaxation: We drop the integrality restriction on all the nonbasic variables  $x_j$ ,  $j \in J$ . We are left with a system of the form

$$\begin{aligned} x &= f + \sum_{j=1}^k r^j s_j \\ x &\in \mathbb{Z}^q \\ s &\geq 0 \end{aligned} \tag{2}$$

where we now denote by  $s$  the nonbasic variables and by  $x$  the remaining basic variables. We will keep this notation in the remainder of the paper. This relaxation of (IP) is denoted by  $R_f(r^1, \dots, r^k)$  where  $f, r^1, \dots, r^k \in \mathbb{Q}^q$ . Such a relaxation was considered in [1] for the case  $q = 2$ . The main result of [1] is a characterization of the polyhedron  $\text{conv}(R_f(r^1, \dots, r^k))$  by split inequalities and intersection cuts arising from triangles and quadrilaterals.

Gomory and Johnson [14] suggested relaxing the  $k$ -dimensional space of variables  $s = (s_1, \dots, s_k)$  to an infinite-dimensional space, where the variables  $s_r$  are defined for any  $r \in \mathbb{Q}^q$ . We get the *semi-infinite relaxation*  $R_f$

$$\begin{aligned} x &= f + \sum r s_r \\ x &\in \mathbb{Z}^q \\ s &\geq 0 \text{ has a finite support.} \end{aligned} \tag{3}$$

By  $s \geq 0$  has a finite support we mean that  $s_r > 0$  for a finite number of  $r \in \mathbb{Q}^q$ . A *feasible solution* of  $R_f$  is a vector  $(x, s)$  that satisfies the three conditions (3). We say that a linear inequality is *valid* for  $R_f$  if it is satisfied by all its feasible solutions. The polyhedron  $\text{conv}(R_f(r^1, \dots, r^k))$  is the face of  $\text{conv}(R_f)$  obtained by setting  $s_r = 0$  for all  $r \in \mathbb{Q}^q \setminus \{r^1, \dots, r^k\}$ . Any valid inequality for (3) yields a valid inequality for (2), and thus for (IP) as well, by simply restricting it to the space  $r^1, \dots, r^k$ .

In the remainder, we assume  $f \notin \mathbb{Z}^q$ . Thus the basic solution  $x = f$ ,  $s = 0$  is not feasible for  $R_f$ . In this paper, we study valid linear inequalities for  $R_f$  that cut off this infeasible basic solution. These inequalities can be stated in terms of the variables  $s$  only:

$$\sum \psi(r)s_r \geq 1 \quad (4)$$

where  $\psi : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$ . Since we are only interested in solutions  $s$  with a finite support, we assume in this paper that the sum in (4) is only taken over those vectors  $r$  such that  $s_r > 0$ . In particular, for any vector  $r$  such that  $\psi(r) = +\infty$  and  $s_r = 0$ , the convention is that  $\psi(r)s_r = 0$ .

Our main interest in this paper is in *minimal* valid inequalities for  $R_f$ , namely valid inequalities  $\sum \psi(r)s_r \geq 1$  such that there is no valid inequality  $\sum \psi'(r)s_r \geq 1$  where  $\psi' \leq \psi$  and  $\psi'(r) < \psi(r)$  for at least one  $r \in \mathbb{Q}^q$ . The reason for this interest is that the linear inequalities needed in any characterization of  $R_f$  are  $s \geq 0$  and minimal valid inequalities. We show that, for a minimal valid inequality  $\sum \psi(r)s_r \geq 1$ , the function  $\psi$  is nonnegative, piecewise linear, convex, and positively homogeneous (namely  $\psi(\lambda r) = \lambda \psi(r)$  for any scalar  $\lambda \in \mathbb{Q}_+$  and  $r \in \mathbb{Q}^q$ ). The function  $\psi$  is not always continuous or finite. However, when it is finite, we show that the piecewise linear function  $\psi$  has at most  $2^q$  pieces.

We make use of the following theorem of Lovász about maximal lattice-free convex sets. A convex set  $S$  is *lattice-free* if it does not have any integral point in its interior. However  $S$  may contain integral points on its boundary.

**Theorem 1.1.** (Lovász [16]) *A maximal lattice-free convex set in  $\mathbb{R}^n$  is either an irrational hyperplane (i.e. it cannot be written in the form  $\alpha_1 x_1 + \dots + \alpha_n x_n = \beta$  with  $\alpha \in \mathbb{Q}^n$ ), or a full-dimensional polyhedron  $P$  of the form  $P = K + L$  where  $K$  is a polytope with  $1 \leq \dim K \leq n$ , and  $L$  is a rational linear space (i.e.  $L$  is generated by rational vectors  $v^1, \dots, v^t \in \mathbb{Q}^n$ , where  $t + \dim K = n$ ). Every facet of  $P$  contains an integral point in its relative interior.*

When  $t \geq 1$  in Theorem 1.1, the polyhedron  $P$  is called a *cylinder over  $K$* .

A theorem of Doignon [10], Bell [5] and Scarf [19] implies that maximal lattice-free convex sets in  $\mathbb{R}^n$  are polyhedra with at most  $2^n$  facets.

Let  $\psi : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$  be a minimal valid function for  $R_f$ . Define

$$B_\psi := \{x \in \mathbb{Q}^q : \psi(x - f) \leq 1\}. \quad (5)$$

Let  $\text{cl}(B_\psi)$  denote the topological closure of  $B_\psi$  in  $\mathbb{R}^q$ . In this paper we prove the following theorem.

**Theorem 1.2.** *Let  $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$ . A minimal valid function  $\psi$  for  $R_f$  is nonnegative, piecewise linear, positively homogeneous and convex. Furthermore the set  $\text{cl}(B_\psi)$  is a full-dimensional maximal lattice-free convex set containing  $f$ . Conversely, for any full-dimensional maximal lattice-free convex set  $B \subset \mathbb{R}^q$  containing  $f$ , there exists a minimal valid function  $\psi$  for  $R_f$  such that  $\text{cl}(B_\psi) = B$ , and when  $f$  is in the interior of  $B$ , this function is unique.*

The remainder of this paper is dedicated to proving this theorem. This result has been applied recently by Espinoza [11] in a computational study of multi-row cuts for (IP), by Cornuéjols and Margot [7] in characterizing the facets of  $R_f$  and  $R_f(r^1, \dots, r^k)$  when  $q = 2$ , and by Dey and Wolsey [9] in their recent study of the corner polyhedron when  $q = 2$ . See also Zambelli [20] and Basu, Bonami, Cornuéjols and Margot [3].

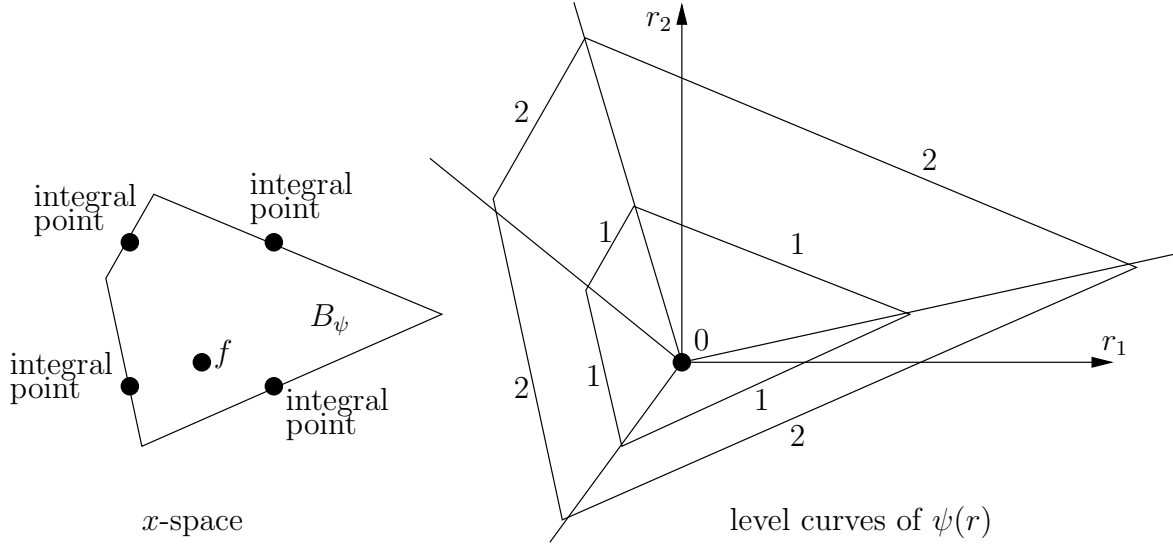


Figure 1: A maximal lattice-free convex set and corresponding function  $\psi$  in  $\mathbb{R}^2$

## 2 Minimal Valid Inequalities

Let  $f \in \mathbb{Q}^q$ . We consider the semi-infinite integer programming problem  $R_f$  defined by (3) where we assume  $f \notin \mathbb{Z}^q$ . Note that  $R_f \neq \emptyset$  since  $x = 0$ ,  $s_r = 1$  for  $r = -f$  and  $s_r = 0$  otherwise, is a feasible solution of (3). Any valid inequality for  $R_f$  that cuts off the infeasible solution  $x = f, s = 0$  can be written as

$$\sum \psi(r) s_r \geq 1. \quad (6)$$

We say that the function  $\psi : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$  is *valid* if the corresponding inequality (6) is satisfied by every feasible solution of  $R_f$ , i.e. by every  $s \geq 0$  with a finite support such that  $f + \sum r s_r \in \mathbb{Z}^q$ .

We assume that there exists at least one feasible solution of  $R_f$  such that  $\sum \psi(r) s_r < +\infty$  (otherwise the function  $\psi$  is uninteresting). When  $\psi(r) = +\infty$  and  $s_r = 0$ , we define  $\psi(r) s_r = 0$ .

**Lemma 2.1.** *If the function  $\psi$  is valid, then  $\psi \geq 0$ .*

*Proof.* Suppose  $\psi(\tilde{r}) < 0$  for some  $\tilde{r} = \begin{pmatrix} \frac{p_1}{D} \\ \dots \\ \frac{p_q}{D} \end{pmatrix}$  where  $p_1, \dots, p_q \in \mathbb{Z}$  and  $D \in \mathbb{Z}_+$  is a common denominator.

Let  $(\bar{x}, \bar{s})$  be a feasible solution of  $R_f$  such that  $\sum \psi(r) \bar{s}_r < +\infty$ . Let  $(\tilde{x}, \tilde{s})$  be defined by  $\tilde{s}_{\tilde{r}} := \bar{s}_{\tilde{r}} + MD$  where  $M$  is a positive integer,  $\tilde{s}_r := \bar{s}_r$  for  $r \neq \tilde{r}$ , and  $\tilde{x} := f + \sum r \tilde{s}_r$ .

The point  $(\tilde{x}, \tilde{s})$  is a feasible solution of  $R_f$  since  $\tilde{x} = \bar{x} + MD\tilde{r} \in \mathbb{Z}^q$ . We have  $\sum \psi(r) \tilde{s}_r = \sum \psi(r) \bar{s}_r + \psi(\tilde{r})MD$ . Choose the integer  $M$  large enough, namely  $M > \frac{\sum \psi(r) \bar{s}_r - 1}{D|\psi(\tilde{r})|}$ . Then  $\sum \psi(r) \tilde{s}_r < 1$ , contradicting the fact that  $(\tilde{x}, \tilde{s})$  is feasible.  $\square$

A valid function  $\psi$  is *minimal* if there is no valid function  $\psi'$  such that  $\psi' \leq \psi$  and  $\psi'(r) < \psi(r)$  for at least one  $r \in \mathbb{Q}^q$ .

**Lemma 2.2.** *If  $\psi$  is a minimal valid function, then  $\psi(0) = 0$ .*

*Proof.* If  $(\bar{x}, \bar{s})$  is a feasible solution in  $R_f$ , then so is  $(\bar{x}, \tilde{s})$  defined by  $\tilde{s}_r := \bar{s}_r$  for  $r \neq 0$ , and  $\tilde{s}_0 = 0$ . Therefore, if  $\psi$  is valid, then  $\psi'$  defined by  $\psi'(r) = \psi(r)$  for  $r \neq 0$  and  $\psi'(0) = 0$  is also valid. Since  $\psi$  is minimal, it follows that  $\psi(0) = 0$ .  $\square$

A function  $g : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$  is *subadditive* if  $g(a) + g(b) \geq g(a + b)$  for all  $a, b \in \mathbb{Q}^q$ .

**Lemma 2.3.** *If  $\psi$  is a minimal valid function, then  $\psi$  is subadditive.*

*Proof.* Let  $r^1, r^2 \in \mathbb{Q}^q$ . Define the function  $\psi'$  as follows.

$$\psi'(r) := \begin{cases} \psi(r^1) + \psi(r^2) & \text{if } r = r^1 + r^2 \\ \psi(r) & \text{if } r \neq r^1 + r^2. \end{cases}$$

We will show that  $\psi'$  is valid. Consider any  $(\bar{x}, \bar{s}) \in R_f$ . Define  $(\bar{x}, \tilde{s})$  as follows

$$\tilde{s}_r := \begin{cases} \bar{s}_{r^1} + \bar{s}_{r^1+r^2} & \text{if } r = r^1 \\ \bar{s}_{r^2} + \bar{s}_{r^1+r^2} & \text{if } r = r^2 \\ 0 & \text{if } r = r^1 + r^2 \\ \bar{s}_r & \text{otherwise.} \end{cases}$$

Using the definitions of  $\psi'$  and  $\tilde{s}$ , it is easy to verify that

$$\sum_r \psi'(r) \tilde{s}_r = \sum_r \psi(r) \tilde{s}_r. \quad (7)$$

Furthermore we have  $\bar{x} = f + \sum r \bar{s}_r = f + \sum r \tilde{s}_r$ . Since  $\bar{x} \in \mathbb{Z}^q$  and  $\tilde{s} \geq 0$ , this implies that  $(\bar{x}, \tilde{s}) \in R_f$ .

Since  $\psi$  is valid, this implies  $\sum_r \psi(r) \tilde{s}_r \geq 1$ . Therefore, by (7),  $\sum \psi'(r) \tilde{s}_r \geq 1$ . Thus  $\psi'$  is valid. Since  $\psi$  is minimal, we get  $\psi(r^1) + \psi(r^2) = \psi'(r^1 + r^2) \geq \psi(r^1 + r^2)$ .  $\square$

A function  $g : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$  is *positively homogeneous* if  $g(\lambda a) = \lambda g(a)$  for all  $\lambda \in \mathbb{Q}_+$  and  $a \in \mathbb{Q}^q$ .

**Lemma 2.4.** *If  $\psi$  is a minimal valid function, then  $\psi$  is positively homogeneous.*

*Proof.* Let  $\tilde{r} \in \mathbb{Q}^q$  and  $\lambda \in \mathbb{Q}_+$ . We will show  $\psi(\lambda \tilde{r}) = \lambda \psi(\tilde{r})$ . This holds when  $\lambda = 0$ . Therefore we assume now  $\lambda > 0$ .

Define the function  $\psi'$  as follows.

$$\psi'(r) := \begin{cases} \frac{1}{\lambda} \psi(\lambda \tilde{r}) & \text{if } r = \tilde{r} \\ \psi(r) & \text{otherwise.} \end{cases}$$

We will show that  $\psi'$  is valid. Consider any  $(\bar{x}, \bar{s}) \in R_f$ . Define  $(\bar{x}, \tilde{s})$  as follows

$$\tilde{s}_r := \begin{cases} \bar{s}_{\lambda \tilde{r}} + \frac{1}{\lambda} \bar{s}_{\tilde{r}} & \text{if } r = \tilde{r} \\ 0 & \text{if } r = \tilde{r} \\ \bar{s}_r & \text{otherwise.} \end{cases}$$

Using the definition of  $\psi'$  and  $\tilde{s}$ , it is easy to verify that

$$\sum \psi'(r) \bar{s}_r = \sum_r \psi(r) \tilde{s}_r. \quad (8)$$

Furthermore we have  $\bar{x} = f + \sum r \bar{s}_r = f + \sum r \tilde{s}_r$ . Since  $\bar{x} \in \mathbb{Z}^q$  and  $\tilde{s} \geq 0$ , this implies that  $(\bar{x}, \tilde{s}) \in R_f$ .

Since  $\psi$  is valid, this implies  $\sum_r \psi(r) \tilde{s}_r \geq 1$ . Therefore, by (8),  $\sum \psi'(r) \bar{s}_r \geq 1$ . Thus  $\psi'$  is valid. Since  $\psi$  is minimal, we get

$$\psi(\lambda \tilde{r}) = \lambda \psi'(\tilde{r}) \geq \lambda \psi(\tilde{r}). \quad (9)$$

Setting  $\tilde{v} = \lambda \tilde{r}$  and  $\mu = \frac{1}{\lambda}$ , (9) becomes

$$\psi(\tilde{v}) \geq \frac{1}{\mu} \psi(\mu \tilde{v}). \quad (10)$$

Since (9) holds for every  $\tilde{r} \in \mathbb{Q}^q$  and  $\lambda \in \mathbb{Q}_+$ , (10) holds for every  $\tilde{v} \in \mathbb{Q}^q$  and  $\mu \in \mathbb{Q}_+$ . (9) and (10) imply  $\psi(\lambda \tilde{r}) = \lambda \psi(\tilde{r})$ .  $\square$

**Corollary 2.5.** *If  $\psi$  is a minimal valid function, then  $\psi$  is convex.*

*Proof.* Let  $r^1, r^2 \in \mathbb{Q}^q$  and  $0 < t < 1$  rational. Then, by Lemmas 2.3 and 2.4,

$$t\psi(r^1) + (1-t)\psi(r^2) = \psi(tr^1) + \psi((1-t)r^2) \geq \psi(tr^1 + (1-t)r^2). \quad \square$$

**Lemma 2.6.** *Let  $\psi$  be a nonnegative, positively homogeneous, subadditive function. Then  $\psi$  is valid for  $R_f$  if and only if  $\psi(x - f) \geq 1$  for all  $x \in \mathbb{Z}^q$ .*

*Proof.* If  $\psi(\bar{x} - f) < 1$  for some  $\bar{x} \in \mathbb{Z}^q$ , set  $\bar{s}_r = 1$  for  $r = \bar{x} - f$  and  $\bar{s} = 0$  otherwise. Then  $(\bar{x}, \bar{s}) \in R_f$  and  $\sum \psi(r) \bar{s}_r = \psi(\bar{x} - f) < 1$ , showing that  $\psi$  is not valid for  $R_f$ .

Conversely, assume  $\psi(x - f) \geq 1$  for all  $x \in \mathbb{Z}^q$ . For any  $(x, s) \in R_f$  we have  $\sum r s_r = x - f$ . Thus  $\psi(\sum r s_r) = \psi(x - f)$ . By subadditivity and positive homogeneity of  $\psi$

$$\sum \psi(r) s_r \geq \psi(\sum r s_r) = \psi(x - f) \geq 1.$$

Therefore  $\psi$  is valid for  $R_f$ .  $\square$

**Example 2.7.** *Let  $f \in \mathbb{Q}^q$  and assume  $0 < f_i < 1$  for some  $i = 1, \dots, q$ . Fix such an index  $i$  and let  $r_i$  denote the  $i$ th component of vector  $r \in \mathbb{Q}^q$ . Define  $\psi_i$  as follows.*

$$\psi_i(r) := \begin{cases} \frac{r_i}{1-f_i} & \text{if } r_i \geq 0 \\ -\frac{r_i}{f_i} & \text{if } r_i \leq 0. \end{cases}$$

*The corresponding inequality  $\sum \psi_i(r) s_r \geq 1$  is the Gomory mixed integer cut [12] obtained from row  $i$  of (3). Equivalently, it is the simple split inequality [6] obtained from the disjunction  $x_i \leq 0$  or  $x_i \geq 1$ .*

**Example 2.8.** In  $\mathbb{R}^q$ , suppose  $f_i = \frac{1}{2}$  for  $i = 1, \dots, q$ . Define  $\psi(r) := \frac{2}{q}(|r_1| + \dots + |r_q|)$ . The corresponding inequality  $\sum \psi(r) s_r \geq 1$  is an intersection cut [2] obtained from the octahedron  $\Omega_f$  centered at  $f$  with vertices  $f \pm \frac{q}{2} e^i$ , where  $e^i$  denotes the  $i$ th unit vector.  $\Omega_f$  has  $2^q$  facets, each of which contains a 0,1 point in its center.

As one would expect, a minimal valid inequality may be implied by a linear combination of other minimal valid inequalities. This is the case here. Indeed, the above intersection cut from the octahedron is implied by the  $q$  split inequalities  $\psi_i(r) := 2|r_i|$  for  $i = 1, \dots, q$  (see Example 2.7 above) since  $\psi = \sum_{i=1}^q \frac{1}{q} \psi_i$  and  $\sum \psi_i(r) s_r \geq 1$  for  $i = 1, \dots, q$  imply  $\sum \psi(r) s_r \geq 1$ .

A minimal valid function  $\psi$  may not be continuous nor finite, as shown by the following example.

**Example 2.9.** In  $\mathbb{R}^2$ , let  $f := \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ . Define  $\psi$  as follows.

$$\psi \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} := \begin{cases} r_2 & \text{if } r_2 > 0 \\ 2|r_1| & \text{if } r_2 = 0 \\ +\infty & \text{if } r_2 < 0. \end{cases}$$

It is easy to verify that this function is valid using Lemma 2.6. Indeed  $\psi(x - f) \geq 1$  for all  $x \in \mathbb{Z}^2$  and equality holds when  $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} i \\ 1 \end{pmatrix}$  for  $i \in \mathbb{Z}$ .

Next we prove the minimality of  $\psi$ . Let  $\varphi$  be a minimal valid function such that  $\varphi \leq \psi$ . By Lemma 2.6  $\varphi(x - f) \geq 1$  for all  $x \in \mathbb{Z}^q$ . In particular, for any point  $\bar{x} \in \mathbb{Z}^q$  such that  $\varphi(\bar{x} - f) = 1$ , the inequality  $\varphi \leq \psi$  implies that we also have

$$\varphi(\bar{x} - f) = 1. \tag{11}$$

Applying (11) to  $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and to  $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , and using the positive homogeneity of  $\varphi$ , we get  $\varphi(r) = \psi(r)$  for all  $r \in \mathbb{Q}^2$  such that  $r_2 = 0$ .

Applying (11) to  $\bar{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}$  for  $i \in \mathbb{Z}$  and using the convexity of  $\varphi$ , we get  $\varphi(x - f) = 1$  for all  $x = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$  for  $\alpha \in \mathbb{Q}$ . By positive homogeneity of  $\varphi$ , we get  $\varphi(r) = \psi(r)$  for all  $r \in \mathbb{Q}^2$  such that  $r_2 > 0$ .

Consider now a vector  $r \in \mathbb{Q}^2$  such that  $r_2 < 0$ . By convexity of  $\varphi$  we have  $\frac{1}{2}\psi \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \frac{1}{2}\psi \begin{pmatrix} M - r_1 \\ -r_2 \end{pmatrix} \geq \psi \begin{pmatrix} \frac{M}{2} \\ 0 \end{pmatrix}$ . Thus  $\psi \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \geq M - |r_2|$ . When  $M$  goes to  $+\infty$ , this implies  $\psi \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = +\infty$ .

Therefore  $\varphi = \psi$ , showing that  $\psi$  is minimal.

### 3 Maximal lattice-free convex sets

Let  $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$  and let  $\psi : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function. Define

$$B_\psi := \{x \in \mathbb{Q}^q : \psi(x - f) \leq 1\}. \tag{12}$$

We are interested in the properties of  $B_\psi$  when  $\psi$  is a minimal valid function for  $R_f$ . For the function  $\psi_i$  of Example 2.7 (the Gomory mixed integer cut),  $B_{\psi_i}$  is the set of rational points satisfying  $0 \leq x_i \leq 1$ . In Example 2.8,  $B_\psi$  is the set of rational points in the octahedron  $\Omega_f$ . For the function  $\psi$  of Example 2.9,  $B_\psi$  contains all the rational points in the band  $0 < x_2 \leq 1$  and in the segment  $x_2 = 0, 0 \leq x_1 \leq 1$ , but the two half-lines  $x_2 = 0, x_1 < 0$ ,  $x_2 = 0, x_1 > 1$  are not in  $B_\psi$ .

For a set  $S \in \mathbb{R}^q$ , let  $\text{cl}(S)$  denote its topological closure. Consider a closed set  $S \in \mathbb{R}^q$ , i.e.  $\text{cl}(S) = S$ . A point  $x \in \mathbb{R}^q$  is in the *interior* of  $S$  if there exists a ball of positive radius centered at  $x$  contained in  $S$ . Thus, if  $S$  has a nonempty interior, it is full-dimensional. The *boundary* of  $S$  is the set of points of  $S$  that are not in its interior. The set  $S$  is said to be *lattice-free* if it contains no integral point in its interior. Note that a lattice-free set may contain integral points on its boundary. We will show that  $\text{cl}(B_\psi)$  is a full-dimensional maximal lattice-free convex set. By Theorem 1.1 this implies that  $\text{cl}(B_\psi)$  is a polyhedron, and by a theorem of Doignon [10], Bell [5] and Scarf [19], it follows that  $\text{cl}(B_\psi)$  has at most  $2^q$  facets.

**Lemma 3.1.** *Let  $\psi$  be a minimal valid function for  $R_f$ . Then  $\text{cl}(B_\psi)$  is a lattice-free convex set in  $\mathbb{R}^q$ . Furthermore  $f \in B_\psi$  and, if  $\psi$  is finite over  $\mathbb{Q}^q$ , then  $\text{cl}(B_\psi)$  contains  $f$  in its interior.*

*Proof.* The fact that  $B_\psi$  is convex over  $\mathbb{Q}^q$  follows from Corollary 2.5 and Definition (12). Therefore  $\text{cl}(B_\psi)$  is convex over  $\mathbb{R}^q$ .

$f \in B_\psi$  since  $\psi(0) = 0$ .

Let  $\bar{x} \in \mathbb{Z}^q$ . By Lemma 2.6,  $\psi(\bar{x} - f) \geq 1$ . If, in addition,  $\bar{x} \in B_\psi$ , this implies  $\psi(\bar{x} - f) = 1$ . Suppose  $\bar{x}$  is in the interior of  $\text{cl}(B_\psi)$ . Then, starting from  $f$  on the straight line passing through  $\bar{x}$ , there exists a point  $\bar{\bar{x}}$  that is beyond  $\bar{x}$  but still in  $B_\psi$ . By positive homogeneity, we have  $\psi(\bar{\bar{x}} - f) > 1$ , which contradicts  $\bar{\bar{x}} \in B_\psi$ . Therefore  $\text{cl}(B_\psi)$  is lattice-free.

Now assume  $\psi(r) < +\infty$  for all  $r \in \mathbb{Q}^q$ . For each unit vector  $e^i$ , the positive homogeneity of  $\psi$  implies the existence of  $\lambda_i > 0$  such that  $\psi(\lambda_i e^i) \leq 1$ . Similarly, for  $i = 1, \dots, q$ , choose any  $\mu_i > 0$  such that  $\psi(\mu_i (-e^i)) \leq 1$ . Then the convex hull of the  $2q$  points  $f + \lambda_i e^i, f - \mu_i e^i$  is contained in  $\text{cl}(B_\psi)$  by convexity and it contains  $f$  in its interior. Thus  $f$  is in the interior of  $\text{cl}(B_\psi)$ .  $\square$

**Lemma 3.2.** *Let  $\psi$  and  $\psi'$  be two convex functions from  $\mathbb{Q}^q$  to  $\mathbb{R} \cup \{+\infty\}$ . Then  $\psi \leq \psi'$  if and only if  $B_{\psi'} \subseteq B_\psi$ . Furthermore if the inclusion  $\text{cl}(B_{\psi'}) \subset \text{cl}(B_\psi)$  is strict, there exists  $r \in \mathbb{Q}^q$  such that  $\psi(r) < \psi'(r)$ .*

*Proof.* The first statement is immediate from the definition (12) of  $B_\psi$ . Assume the inclusion  $\text{cl}(B_{\psi'}) \subset \text{cl}(B_\psi)$  is strict. Then there exists  $\bar{x} \in B_\psi \setminus B_{\psi'}$ . This implies  $\psi(r) \leq 1 < \psi'(r)$  for  $r = f - \bar{x}$ .  $\square$

### 3.1 Maximal lattice-free convex sets with $f$ in the interior

**Definition 3.3.** *Let  $B$  be a full-dimensional maximal lattice-free convex set in  $\mathbb{R}^q$  and let  $f \in \mathbb{Q}^q$  be a point in its interior.  $B$  is a polyhedron by Theorem 1.1. The recession cone of  $B$  is the set of vectors  $r \in \mathbb{R}^q$  such that the whole ray  $f + \lambda r, \lambda \geq 0$ , is contained in  $B$ .*

Define the function  $\psi_B : \mathbb{Q}^q \rightarrow \mathbb{R}$  as follows. Set  $\psi_B(r) = 0$  for any vector  $r \in \mathbb{Q}^q$  in the recession cone of  $B$ . For any  $r \in \mathbb{Q}^q$  that is not in the recession cone of  $B$ , set  $\psi_B(r) = \frac{1}{\lambda}$  where  $\lambda > 0$  is the scalar for which the point  $f + \lambda r$  is on the boundary of  $B$ .

Observe that the function  $\psi_B$  defined in 3.3 is nonnegative, positively homogeneous, and satisfies  $B_{\psi_B} = B \cap \mathbb{Q}^q$ . The next lemma shows that it is a minimal valid function for  $R_f$ . Theorem 3.5 will show that, conversely, all finite minimal valid functions for  $R_f$  are of this form.

**Lemma 3.4.** *Let  $B$  be any full-dimensional maximal lattice-free convex set in  $\mathbb{R}^q$ . If  $B$  contains  $f \in \mathbb{Q}^q$  in its interior, the function  $\psi_B$  defined in 3.3 is a minimal valid function for  $R_f$  such that  $\text{cl}(B_{\psi_B}) = B$ .*

*Proof.* The fact that  $\text{cl}(B_{\psi_B}) = B$  follows from  $B_{\psi_B} = B \cap \mathbb{Q}^q$ . We now show that  $\psi_B$  is a minimal valid function for  $R_f$ .

**Claim 1:**  $\psi_B$  is subadditive.

The proof uses the convexity of  $B$  and the positive homogeneity of  $\psi_B$ .

Let  $a, b \in \mathbb{Q}^q$ . Suppose first that neither  $a$  nor  $b$  is in the recession cone of  $B$ . Set  $\alpha > 0$  to be the scalar such that  $\psi_B(\alpha a) = 1$ . Similarly set  $\beta > 0$  such that  $\psi_B(\beta b) = 1$ . Then  $f + \alpha a, f + \beta b \in B$ . By convexity of  $B$ , for any  $0 \leq \lambda \leq 1$  we have  $f + \lambda \alpha a + (1 - \lambda) \beta b \in B$ . Therefore

$$\lambda \psi_B(\alpha a) + (1 - \lambda) \psi_B(\beta b) = 1 \geq \psi_B(\lambda \alpha a + (1 - \lambda) \beta b). \quad (13)$$

Set  $\lambda := \frac{\beta}{\alpha + \beta}$  in (13). We get, by positive homogeneity of  $\psi_B$

$$\psi_B(a) + \psi_B(b) \geq \psi_B(a + b). \quad (14)$$

If both  $a, b$  are in the recession cone of  $B$ , then  $a + b$  also is and again (14) holds. So we may assume that  $b$  is in the recession cone of  $B$  but not  $a$ . Choose  $\alpha > 0$  such that  $\psi_B(\alpha a) = 1$ . Then  $\alpha a \in B$  and, since  $b$  is in the recession cone of  $B$ , we also have  $\alpha a + \alpha b \in B$ . Thus  $\psi_B(\alpha(a + b)) \leq 1$ . Now (14) holds since  $\psi_B(\alpha a) + \psi_B(\alpha b) = 1$  and  $\psi_B$  is positively homogeneous.

Using Lemma 2.6 and Claim 1, we get that  $\psi_B$  is valid.

**Claim 2:**  $\psi_B$  is minimal.

Suppose not. Let  $\psi$  be a minimal valid function for  $R_f$  such that  $\psi \leq \psi_B$  and  $\psi(\bar{r}) < \psi_B(\bar{r})$  for some  $\bar{r} \in \mathbb{Q}^q$ . Positive homogeneity of  $\psi_B$  and  $\psi$  implies that there exist  $\mu > \lambda > 0$  such that  $\psi_B(\lambda \bar{r}) = 1$  and  $\psi(\mu \bar{r}) \leq 1$ . Let  $\bar{x} = f + \lambda \bar{r}$  and let  $\bar{\bar{x}}$  be a point strictly between  $\bar{x}$  and  $f + \mu \bar{r}$ . Then  $\psi_B(\bar{\bar{x}} - f) > 1$ , which implies  $\bar{\bar{x}} \notin B$ . But  $\psi(\bar{\bar{x}} - f) < 1$ , which implies  $\bar{\bar{x}}$  is in the interior of  $\text{cl}(B_\psi)$ . It follows that  $B$  is strictly contained in  $\text{cl}(B_\psi)$ . By Lemma 3.1,  $\text{cl}(B_\psi)$  is a lattice-free convex set. This contradicts the assumption that  $B$  is a maximal lattice-free convex set. Thus  $\psi_B$  is a minimal valid function for  $R_f$ .  $\square$

**Theorem 3.5.** *Let  $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$ . If  $\psi$  is a finite minimal valid function for  $R_f$ , then  $\psi$  is a nonnegative, positively homogeneous, piecewise linear, convex function with at most  $2^q$  pieces. Furthermore  $\psi$  can be extended to  $\mathbb{R}^q$  into a continuous function.*

*Proof.* Let  $\psi$  be a finite minimal valid function for  $R_f$ . From the results of Section 2, we know that  $\psi$  is nonnegative, positively homogeneous, and convex. We will now show that  $\psi$  is piecewise linear.

Let  $B_\psi$  be the set defined in (12). By Lemma 3.1,  $\text{cl}(B_\psi)$  is a lattice-free convex set, and  $f$  is in its interior. Thus  $\text{cl}(B_\psi)$  is full-dimensional. Suppose  $\text{cl}(B_\psi)$  is not maximal and let  $B$  be a maximal lattice-free convex set such that  $\text{cl}(B_\psi) \subset B$ . Then by Lemma 3.4 there exists a minimal valid function  $\psi_B$  such that  $\text{cl}(B_{\psi_B}) = B$ . By Lemma 3.2,  $\psi_B \leq \psi$  and there exists  $r$  such that  $\psi_B(r) < \psi(r)$ , contradicting the minimality of  $\psi$ . Thus  $\text{cl}(B_\psi)$  is a maximal lattice-free convex set.

By Theorem 1.1 every maximal lattice-free convex set is a polyhedron and by a theorem of Doignon [10], Bell [5] and Scarf [19], this polyhedron has at most  $2^q$  facets. The proof of the upper bound on the number of facets is simple and elegant: By Theorem 1.1, each facet  $F$  contains an integral point  $x^F$  in its relative interior. If there are more than  $2^q$  facets, two integral points  $x^F$  and  $x^{F'}$  must be identical modulo 2. Then their middle point  $\frac{1}{2}(x^F + x^{F'})$  is integral and interior, contradiction.

Consider a facet  $F$  of  $B_\psi$ . We have  $\psi(x - f) = 1$  for all  $x \in F \cap \mathbb{Q}^q$ . By positive homogeneity (Lemma 2.4),  $\psi$  is linear in the cone  $\{r \in \mathbb{Q}^q : r = \lambda(x - f) \text{ with } \lambda \geq 0, x \in F\}$ . Since the union of these cones over all facets of  $B_\psi$  covers  $\mathbb{Q}^q$ , the function  $\psi$  is piecewise linear with at most  $2^q$  pieces.

Since the linear extension of  $\psi$  to  $\mathbb{R}^q$  is continuous in each cone and the values match at the boundary of the cones, the last statement of the theorem follows.  $\square$

Even though a minimal valid function  $\psi$  is only defined over rational vectors (and  $f$  is a rational vector), the facets of the corresponding polyhedron  $\text{cl}(B_\psi)$  may be defined by irrational hyperplanes [7].

### 3.2 Maximal lattice-free convex sets with $f$ on the boundary

To complete the proof of Theorem 1.2, we consider the case of a minimal valid function  $\psi$  that is not finite everywhere, i.e.  $\psi(r) = +\infty$  for some  $r \in \mathbb{Q}^q$ . This corresponds to a set  $B_\psi$  where  $f$  is on the boundary of  $\text{cl}(B_\psi)$ . One could argue that this case is unimportant from a practical point of view since Zambelli [20] showed recently that every minimal valid inequality for  $R_f(r^1, \dots, r^k)$  can be derived from a function  $\psi$  where  $f$  is in the interior of  $\text{cl}(B_\psi)$ . The minimal valid inequalities for  $R_f$  are more complicated. We present the result for the sake of completeness.

In this section, we need to define lattice-free convex sets in affine subspaces of  $\mathbb{R}^q$ . A point  $x \in \mathbb{R}^q$  is in the *relative interior* of a convex  $S$  if there exists a ball  $B$  of positive radius centered at  $x$  such that  $B \cap \text{aff}(S)$  is contained in  $S$ , where  $\text{aff}(S)$  denotes the affine hull of  $S$ . The *boundary* of  $S$  is the set of points of  $S$  that are not in its relative interior. The set  $S$  is said to be *lattice-free* if it contains no integral point in its relative interior.

**Definition 3.6.** Let  $B$  be a full-dimensional maximal lattice-free convex set in  $\mathbb{R}^q$  and let  $f \in \mathbb{Q}^q$  be a point on the boundary of  $B$ .  $B$  is a polyhedron by Theorem 1.1. Let  $\mathcal{F}$  denote the family of all faces of  $B$  that contain  $f$  and have dimension at least one. In particular,  $B \in \mathcal{F}$ . Define  $M_B := B$ . We will define sets  $M_F$  for  $F \in \mathcal{F} \setminus \{B\}$  starting from faces  $F$  of dimension  $\dim(B) - 1$  and then recursively decreasing the dimension by 1. For each

$F \in \mathcal{F} \setminus \{B\}$ , let  $\mathcal{F}_F$  denote the set of all faces  $G \neq F$  of  $B$  that contain the face  $F$ , and let  $M_F$  be a maximal lattice-free convex subset of  $F \cap (\bigcap_{G \in \mathcal{F}_F} M_G)$  that contains  $f$ . A slight generalization of Theorem 1.1 shows that a maximal lattice-free convex subset of a polyhedron is a polyhedron (this is easy to show when the polyhedron is rational; see [4] for a proof in the general case). Thus  $M_F$  is a polyhedron.

Define the function  $\psi_{\mathcal{F}} : \mathbb{Q}^q \rightarrow \mathbb{R} \cup \{+\infty\}$  as follows.  $\psi_{\mathcal{F}}(0) = 0$ . For each  $r \in \mathbb{Q}^q \setminus \{0\}$ , let  $F \in \mathcal{F}$  be the face of lowest dimension such that the ray  $R := \{x = f + \lambda r : \lambda \geq 0\}$  goes through the relative interior of  $F$ . If there is no such face or if  $R \cap M_F = \{f\}$ , set  $\psi_{\mathcal{F}}(r) = +\infty$ . If  $r$  is in the recession cone of  $M_F$ , set  $\psi_{\mathcal{F}}(r) = 0$ . Otherwise, set  $\psi_{\mathcal{F}}(r) = \frac{1}{\lambda}$  where  $\lambda$  is the positive scalar for which the point  $f + \lambda r$  is on the boundary of  $M_F$ .

**Example 3.7.** If  $B$  is the set  $0 \leq x_2 \leq 1$  in  $\mathbb{R}^2$ , and  $f := \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ , then  $\mathcal{F}$  consists of  $B$  and its face  $F$  defined by the line  $x_2 = 0$ . The set  $M_F$  is the segment  $0 \leq x_1 \leq 1, x_2 = 0$ , which is the unique maximal lattice-free convex set contained in  $F$  that contains  $f$ . The corresponding function  $\psi_{\mathcal{F}}$  is the function  $\psi$  of Example 2.9.

**Lemma 3.8.** The function  $\psi_{\mathcal{F}}$  defined in 3.6 is a valid function for  $R_f$ .

*Proof.* **Claim:**  $\psi_{\mathcal{F}}$  is subadditive.

Let  $a, b \in \mathbb{Q}^q$ . The result holds if  $\psi_{\mathcal{F}}(a) = +\infty$  or  $\psi_{\mathcal{F}}(b) = +\infty$ , so we assume  $\psi_{\mathcal{F}}(a) < +\infty$  and  $\psi_{\mathcal{F}}(b) < +\infty$ . Let  $F_a \in \mathcal{F}$  be the face of lowest dimension such that the ray  $R_a := \{x = f + \lambda a : \lambda \geq 0\}$  goes through the relative interior of  $F_a$ . Define  $F_b$  and  $R_b$  similarly, as well as  $F_{a+b}$  and  $R_{a+b}$ . Note that both  $F_a$  and  $F_b$  are faces of  $F_{a+b}$ .

Suppose first that neither  $a$  is in the recession cone of  $M_{F_a}$  nor  $b$  is in the recession cone of  $M_{F_b}$ . Let  $\alpha > 0$  and  $\beta > 0$  be such that  $\psi_{\mathcal{F}}(\alpha a) = 1$  and  $\psi_{\mathcal{F}}(\beta b) = 1$  respectively. Thus  $f + \alpha a \in M_{F_a}$  and  $f + \beta b \in M_{F_b}$ . Since  $M_{F_{a+b}}$  is a convex set and  $M_{F_a} \subseteq M_{F_{a+b}}$ ,  $M_{F_b} \subseteq M_{F_{a+b}}$ , it follows that, for any  $0 \leq \lambda \leq 1$  the point  $\lambda(f + \alpha a) + (1 - \lambda)(f + \beta b)$  belongs to  $M_{F_{a+b}}$ . Therefore  $\psi_{\mathcal{F}}(\lambda \alpha a + (1 - \lambda)\beta b) \leq 1$ . Thus we have

$$\lambda \psi_{\mathcal{F}}(\alpha a) + (1 - \lambda) \psi_{\mathcal{F}}(\beta b) = 1 \geq \psi_{\mathcal{F}}(\lambda \alpha a + (1 - \lambda)\beta b). \quad (15)$$

Set  $\lambda := \frac{\beta}{\alpha + \beta}$  in (15). We get, by positive homogeneity of  $\psi_{\mathcal{F}}$

$$\psi_{\mathcal{F}}(a) + \psi_{\mathcal{F}}(b) \geq \psi_{\mathcal{F}}(a + b). \quad (16)$$

If both  $a$  is in the recession cone of  $M_{F_a}$  and  $b$  is in the recession cone of  $M_{F_b}$ , then  $a + b$  is in the recession cone of  $M_{F_{a+b}}$  and again (16) holds. So we may assume that  $b$  is in the recession cone of  $M_{F_b}$  and that  $a$  is not in the recession cone of  $M_{F_a}$ . Choose  $\alpha > 0$  such that  $\psi_{\mathcal{F}}(\alpha a) = 1$ . Then  $\alpha a \in M_{F_a}$  and, since  $b$  is in the recession cone of  $M_{F_b}$ , we have  $\alpha a + \alpha b \in M_{F_{a+b}}$ . Thus  $\psi_{\mathcal{F}}(\alpha(a + b)) \leq 1$ . Now (16) holds since  $\psi_{\mathcal{F}}(\alpha a) + \psi_{\mathcal{F}}(\alpha b) = 1$  and  $\psi_{\mathcal{F}}$  is positively homogeneous. This proves the claim.

Lemma 2.6 and the construction of Definition 3.6 imply that  $\psi_{\mathcal{F}}$  is valid.  $\square$

**Theorem 3.9.** If  $\psi$  is a minimal valid function for  $R_f$ , then  $\psi$  is a nonnegative, piecewise linear, positively homogeneous, convex function.

*Proof.* Let  $B_\psi$  be the corresponding convex set as defined in (12). By Lemma 3.1  $\text{cl}(B_\psi)$  is a lattice-free convex set in  $\mathbb{R}^q$  and  $f \in B_\psi$ .

First we prove that  $\text{cl}(B_\psi)$  is a maximal lattice-free convex set in  $\mathbb{R}^q$ . Suppose not and let  $B$  be a maximal lattice-free convex set that contains  $\text{cl}(B_\psi)$ . Then there is a point  $b$  in the relative interior of  $B$  that is not in  $\text{cl}(B_\psi)$ . Let  $D := \text{conv}(\text{cl}(B_\psi) \cup \{b\})$ . Define  $\psi_D$  as follows. For any  $r \in \mathbb{Q}^q$  such that the ray  $R_r := \{x = f + \lambda r : \lambda \geq 0\}$ , does not go through the relative interior of  $D$ , define  $\psi_D(r) = \psi(r)$ . If the ray  $R_r$  goes through the relative interior of  $D$  and is in the recession cone of  $D$ , set  $\psi_D(r) = 0$ . Otherwise let  $\psi_D(r) = \frac{1}{\lambda}$  where  $\lambda > 0$  is the scalar such that the point  $f + \lambda r$  is on the boundary of  $D$ . The function  $\psi_D$  is nonnegative and positively homogeneous. Next we show that  $\psi_D$  is subadditive. Let  $a, b \in \mathbb{Q}^q$ . If neither  $R_a, R_b$  goes through the relative interior of  $D$ ,  $\psi_D(a) + \psi_D(b) = \psi(a) + \psi(b) \geq \psi(a+b) \geq \psi_D(a+b)$ . If both  $R_a, R_b$  go through the relative interior of  $D$ , the convexity of  $D$  and positive homogeneity of  $\psi_D$  imply the subadditivity of  $\psi_D$ . Similarly when  $R_a$  goes through the relative interior of  $D$  but  $R_b$  does not. Thus  $\text{cl}(B_\psi)$  is a maximal lattice-free convex set in  $\mathbb{R}^q$ .

By Theorem 1.1,  $\text{cl}(B_\psi)$  is a polyhedron. Suppose first that  $\text{cl}(B_\psi)$  is a hyperplane. Since  $B_\psi \in \mathbb{Q}^q$ , this implies that  $\text{cl}(B_\psi)$  is a rational hyperplane, contradicting Theorem 1.1. Therefore  $\text{cl}(B_\psi)$  is a full-dimensional polyhedron. If  $f$  is in the interior of  $\text{cl}(B_\psi)$ , the theorem follows from the results in Section 3.1. Therefore we assume now that the point  $f$  lies on the boundary of  $\text{cl}(B_\psi)$ . Let  $\mathcal{F}$  be the family of faces of  $\text{cl}(B_\psi)$  that contain  $f$  and have dimension at least one. For each  $F \in \mathcal{F}$ , define

$$\Phi_F := \text{cl}(\{x \in \mathbb{Q}^q \cap F : \psi(x - f) \leq 1\}).$$

Let  $\mathcal{F}_F$  denote the set of all faces  $G \neq F$  of  $B$  that contain  $F$ . Note that  $\Phi_F \subseteq \bigcap_{G \in \mathcal{F}_F} \Phi_G$  since  $\psi$  is a convex function. Furthermore  $\Phi_F$  is a lattice-free convex subset of  $F$ . We claim that  $\Phi_F$  is a maximal lattice-free convex subset of  $F \cap (\bigcap_{G \in \mathcal{F}_F} \Phi_G)$  for all  $F \in \mathcal{F}$ . Suppose not. Then there exist maximal lattice-free convex sets  $M_F \subseteq F \cap (\bigcap_{G \in \mathcal{F}_F} M_G)$  such that  $\Phi_F \subseteq M_F$  for all  $F \in \mathcal{F}$  and the inclusion  $\Phi_F \subseteq M_F$  is strict for at least one face  $F \in \mathcal{F}$ . Thus the function  $\psi_{\mathcal{F}}$  defined in 3.6 satisfies  $\psi_{\mathcal{F}} \leq \psi$  and  $\psi_{\mathcal{F}}(r) < \psi(r)$  for at least one  $r \in \mathbb{Q}^q$ . By Lemma 3.8,  $\psi_{\mathcal{F}}$  is valid for  $R_f$ , but this contradicts the minimality of  $\psi$ . Thus  $\Phi_F$  is a maximal lattice-free convex subset of  $F$  for all  $F \in \mathcal{F}$ . These sets are polyhedra [4]. Thus  $\psi$  is a piecewise linear function.  $\square$

## 4 Conclusion

Most cutting planes used in integer programming can be viewed in the context of Gomory's corner polyhedron. A natural first step in studying the corner polyhedron is to investigate the semi-infinite relaxation  $R_f$ . This paper establishes a close connection between minimal valid inequalities for  $R_f$  and maximal lattice-free convex sets in the space of the integer variables. We use this connection to prove that minimal valid inequalities for  $R_f$  are nonnegative, positively homogeneous, piecewise linear, convex functions. Interest in cutting planes from two rows of a simplex tableau was initiated by Dey and Richard [8], Andersen, Louveaux, Weismantel and Wolsey [1], and Gomory [15]. Our results on minimal valid inequalities for  $R_f$  have been used in several recent publications (Espinoza [11], Dey and Wolsey [9], Cornuéjols and Margot [7], Zambelli [20]). Further investigations are under way.

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## References

- [1] K. Andersen, Q. Louveaux, R. Weismantel and L. Wolsey, Cutting Planes from Two Rows of a Simplex Tableau, *Proceedings of IPCO XII*, Ithaca, New York (2007) 1-15.
- [2] E. Balas, Intersection Cuts - A New Type of Cutting Planes for Integer Programming, *Operations Research* 19 (1971) 19-39.
- [3] A. Basu, P. Bonami, G. Cornuéjols and F. Margot, On the Relative Strength of Split, Triangle and Quadrilateral Cuts, technical report, July 2008.
- [4] M. Conforti, G. Cornuéjols and G. Zambelli, Maximal lattice-free convex subsets of linear spaces, technical report, August 2008.
- [5] D.E. Bell, A Theorem Concerning the Integer Lattice, *Studies in Applied Mathematics* 56 (1977) 187-188.
- [6] W. Cook, R. Kannan and A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming* 47 (1990) 155-174.
- [7] G. Cornuéjols and F. Margot, On the facets of Mixed Integer Programs with Two Integer Variables and Two Constraints, to appear in *Mathematical Programming* (published online May 2008), <http://dx.doi.org/10.1007/s10107-008-0221-1>.
- [8] S.S. Dey and J.-P. P. Richard, Facets for the Two-Dimensional Infinite Group Problems, *Mathematics of Operations Research* 33 (2008) 140-166.
- [9] S.S. Dey and L.A. Wolsey, Lifting Integer Variables in Minimal Inequalities Corresponding to Lattice-Free Triangles, *IPCO 2008*, Bertinoro, Italy (May 2008), *LNCS 5035* 463-475.
- [10] J.-P. Doignon, Convexity in Cristallographical Lattices, *Journal of Geometry* 3 (1973) 71-85.
- [11] D. Espinoza, Computing with multi-row Gomory cuts, *IPCO 2008*, Bertinoro, Italy (May 2008), *LNCS 5035* 214-224.
- [12] R.E. Gomory, An Algorithm for Integer Solutions to Linear Programs, *Recent Advances in Mathematical Programming*, R.L. Graves and P. Wolfe eds., McGraw-Hill, New York (1963) 269-302.
- [13] R.G. Gomory, Some polyhedra related to combinatorial problems, *Linear Algebra and Applications* 2 (1969) 451-558.
- [14] R.E. Gomory and E.L. Johnson, Some Continuous Functions Related to Corner Polyhedra, Part I, *Mathematical Programming* 3 (1972) 23-85.

- [15] R.E. Gomory, Thoughts about Integer Programming, 50th Anniversary Symposium of OR, University of Montreal, January 2007, and Corner Polyhedra and Two-Equation Cutting Planes, George Nemhauser Symposium, Atlanta, July 2007.
- [16] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developments and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989) 177-201.
- [17] H. Marchand and L.A. Wolsey, Aggregation and mixed integer rounding to solve MIPs, *Operations Research* 49 (2001) 363–371.
- [18] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley (1988).
- [19] H.E. Scarf, An Observation on the Structure of Production Sets with Indivisibilities, *Proceedings of the National Academy of Sciences of the United States of America* 74 (1977) 3637-3641.
- [20] G. Zambelli, On Degenerate Multi-row Gomory Cuts, technical report, April 2008, to appear in *Operations Research Letters*.